

Chapter 7. Techniques of Integration

Section 7.1 - Integration By Parts.

Every differentiation rule has a corresponding Integration rule. For example, the substitution rule corresponds to the chain rule. The rule that corresponds to the product rule is called "Integration by Parts".

Recall the product rule: $\frac{d}{dx} (f(x) \cdot g(x)) = f'(x)g(x) + f(x)g'(x)$.

Taking integrals on both sides:

$$\rightarrow f(x) \cdot g(x) = \int f'(x)g(x) dx + \int f(x)g'(x) dx.$$

Rearrange this equation as:

$$\boxed{\int f(x)g'(x) dx = f(x)g(x) - \int g(x)f'(x) dx.}$$

This is called the "Integration by Parts" formula. An alternative Version is given by $\boxed{\int u dv = uv - \int v du}$

Here $u=f(x)$, $v=g(x)$, $du=f'(x)dx$ and $dv=g'(x)dx$.

Examples. ① Find $\int x \cos x dx$. Let $u=x$ and $dv=\cos x dx$. Then $du=dx$ and $v=\sin x$.

$$\text{Then } \int x \cos x dx = x \cdot \sin x - \int \sin x \cdot dx = x \sin x + \cos x + C.$$

Note: the choice of $u=\cos x$ and $dv=x dx$ yields

$$du=-\sin x dx \text{ and } v=\frac{x^2}{2}, \text{ and so}$$

$$\int x \cos x = \frac{x^2}{2} \cos x - \int -\frac{x^2}{2} \sin x dx \leftarrow \text{Harder than original Integral!}$$

② Find $\int \ln x \, dx$. Let $u = \ln x$, $dv = dx$; So $du = \frac{1}{x} dx$, $v = x$, and so
 $\int \ln x \, dx = x \cdot \ln x - \int x \cdot \frac{1}{x} \, dx = x \cdot \ln x - \int 1 \, dx = x \ln x - x + C$.

③ $\int t^2 e^{-t} \, dt$. Let $u = t^2$, $dv = e^{-t} \, dt \Rightarrow du = 2t \, dt$, $v = -e^{-t}$. So
 $\int t^2 e^{-t} \, dt = -t^2 e^{-t} + \int 2t e^{-t} \, dt$. Let $u = 2t$, $dv = e^{-t} \, dt$,
 $du = 2 \, dt$, $v = -e^{-t}$. So
 $\int t^2 e^{-t} \, dt = -t^2 e^{-t} + 2t(-e^{-t}) - \int -2e^{-t} \, dt$
 $= -t^2 e^{-t} - 2t e^{-t} - 2e^{-t} + C = -e^{-t}(t^2 + 2t + 2) + C$.

④ $\int e^x \cos x \, dx$. Let $u = e^x$, $dv = \cos x \, dx \Rightarrow du = e^x \, dx$, $v = \sin x$.
 $A = \int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$. Let $u = e^x$, $dv = \sin x \, dx$
 $A = e^x \sin x - (-e^x \cos x - \int -e^x \cos x \, dx)$ $\quad du = e^x \, dx$, $v = -\cos x$.
 $A = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx = e^x(\sin x + \cos x) - A$.
 $\rightarrow 2A = e^x \sin x + e^x \cos x + C \Rightarrow A = \frac{1}{2} e^x (\sin x + \cos x) + C$.

⑤ $\int \tan^{-1}(x) \, dx$. Let $u = \tan^{-1}(x)$, $dv = dx$; So $du = \frac{1}{1+x^2} \, dx$, $v = x$. Then
 $\int \tan^{-1}(x) \, dx = x \cdot \tan^{-1}(x) - \int \frac{x}{1+x^2} \, dx \leftarrow u\text{-sub}, u = 1+x^2$.
 $\frac{du}{2} = x \, dx$.
 $\int \tan^{-1}(x) \, dx = x \cdot \tan^{-1}(x) - \frac{1}{2} \ln |1+x^2| + C$.

Evaluating Definite Integrals. $\int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du$.

Example ① $\int_0^1 \sin^{-1}(x) \, dx$. Let $u = \sin^{-1}(x)$, $dv = dx \Rightarrow du = \frac{1}{\sqrt{1-x^2}} \, dx$, $v = x$.

$$\int_0^1 \sin^{-1}(x) \, dx = \left. x \cdot \sin^{-1}(x) \right|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} \, dx.$$

let $w = 1-x^2$, $-\frac{dw}{2} = x \, dx$. Then

$$\begin{aligned}\int_0^1 \sin^{-1}(x) \, dx &= x \cdot \sin^{-1}(x) \Big|_0^1 - \left(-\frac{1}{2} \int \frac{dw}{\sqrt{w}} \right) = x \sin^{-1}(x) \Big|_0^1 + \sqrt{w} \\ &= x \cdot \sin^{-1}(x) \Big|_0^1 + \sqrt{1-x^2} \Big|_0^1 = \sin^{-1}(1) - \sqrt{1} = \frac{\pi}{2} - 1.\end{aligned}$$

Example with root-exponentials. Find $\int e^{\sqrt{2s+6}} \, ds$

let $x = \sqrt{2s+6}$, then $dx = \frac{1}{\sqrt{2s+6}} \, ds = \frac{1}{x} \, ds \Rightarrow ds = x \cdot dx$. Then

$$\int e^{\sqrt{2s+6}} \, ds = \int x e^x \, dx \rightarrow \text{By Part: let } u=x, dv=e^x \, dx \rightarrow du=dx, v=e^x.$$

$$\text{So, } \int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C. \text{ Therefore, we have}$$

$$\int e^{\sqrt{2s+6}} \, ds = e^{\sqrt{2s+6}} (\sqrt{2s+6} - 1) + C.$$